Aug 29, 2014 Shanghai

Semisimple Hopf actions on ~~quantizations~~ Weyl algebras

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$k = \bar{k}$, $\mathrm{ch}\, k = 0$.

Big Goal To understand (examples of) Hopf alg actions on algebras

Given Hopf alg H, alg A, one could

(what we'll do here) $\boxed{\text{Show that } H \curvearrowright A \text{ factors through group action}}$ (no quantum symmetry)

classify all $H \curvearrowright A$ (possibly lots of quantum symmetry)

Hard because { computations can be horrible; depends on classification of Hopf algs.

Defn H acts on A if A is a (left) H-module algebra $\Leftrightarrow$ A is an algebra in H-mod
(A = (left) H-module, m_A & u_A are H-maps)

If H finite dim'l, there are two tractable classes of H

(here) $\boxed{\text{semisimple}}$ (as an k-alg)

pointed (all simple H-comodules are 1-dim'l)

Have a nice result when A is a commutative domain

(*) Thm [Etingof–W] Any semisimple Hopf action on a commutative domain factors through a group algebra action.

(In other words, if $\underset{ss}{H} \curvearrowright A$ inner-faithfully ($\nexists I^{\neq 0}$ Hopf ideal of H $\ni$ there's an induced action of H/I on A)
then $H \cong kG$ group algebra)

We extend this result to semisimple Hopf alg actions on quantizations of commutative domains B

Take $B = A_n(k)$ (Weyl algebra) $= k\langle x_i, y_i\rangle_{i=1}^{n} \Big/ \left(\begin{matrix}[x_i,x_j]=[y_i,y_j]=0\\ [y_i,x_j]=\delta_{ij}\end{matrix}\right)$

Thm [Cuadra-Etingof-W] Any semisimple Hopf action on $A_n(k)$ factors through a group algebra action.

no longer commutative, proof is different from (*)

[Chan-Wang-W-Zhang] proved this result when

- $A_n(k)$ has the standard filtration
- $H \curvearrowright A_n(k)$ preserves the filtration of $A_n(k)$

we don't assume this

Proof follows in two steps: ① Reduce $H \curvearrowright B = A_n(k)$ modulo p (so that B is a PI domain & can localize)

② Study $H_p \curvearrowright$ division alg in positive char.

Step ② Prop H = semisimple, cosemisimple Hopf algebra of dimension d, over an algebraically closed field F of arbitrary characteristic.

D = division algebra over F of degree N over $Z(D)$

Then $\gcd(d!, N) = 1 \Rightarrow H \curvearrowright D$ factors through a group action.

Pf/ degree argument with $\gcd(d!,N)=1 \rightsquigarrow D = ZD^H$ (a subalgebra of D), $Z = Z(D)$

$\Rightarrow C_D(D^H) = Z$

$\Rightarrow Z$ is H-stable ($H \curvearrowright Z$) [can show $\forall z\in Z,\ h\in H,\ a\in D^H$: $(h\cdot z)a = a(h\cdot z)$]

Assume $H \curvearrowright D$ inner-faithfully. Take I = Hopf ideal of H so that $I\cdot Z = 0$

$D = ZD^H \Rightarrow I\cdot D = 0 \xRightarrow{\text{inner faithfulness}} I = (0)$

$\Rightarrow H \curvearrowright Z$ (a field) inner-faithfully

$\xRightarrow{\text{Thm in [EW]}}$ H is a group algebra. ///

Step ① Reduction modulo p

- Given $H \curvearrowright B$ inner faithful

$\boxed{\text{Goal}}$: show for prime # $p \gg 0$, $\exists$ action $H_p \curvearrowright B_p$ where H_p (over $\overline{\mathbb{F}_p}$) ss/coss & action is inner faithful.

* Given $H \curvearrowright B = A_n(k)$ $\rightsquigarrow$ $\exists$ f.g. subring R of k so that $H_R \curvearrowright B_R = A_n(R)$
  H_R Hopf R-order in H (R-subalgebra of H)

  Here $\boxed{H \simeq k \otimes_R H_R}$

* $\exists$ homomorphism $\psi : R \to \overline{\mathbb{F}_p}$ for $p \gg 0$ $\rightsquigarrow$ take $\boxed{H_{\psi,p} = H_R \otimes_R \overline{\mathbb{F}_p} =: H_p}$
* Get $\underset{\text{ss\&coss}}{H_p} \curvearrowright B_p =: A_n(\overline{\mathbb{F}_p})$ for $p \gg 0$

* Further, $H_p \curvearrowright B_p$ inner faithful.

<u>Now Proof of theorem</u>: * Assume $H \curvearrowright B = A_n(k)$ is inner faithful
* Reduce modulo p by step ①: $\underset{\text{ss\&coss}}{H_p} \curvearrowright B_p$ inner faithful
  a PI domain $\rightsquigarrow$ <u>can localize</u>

* Take D_p = quotient division ring of B_p.
* [Skryabin–van Oystaeyen] $\Rightarrow$ $H_p \curvearrowright D_p$ inner-faithfully
* Take $p > \dim H$, Have that $\deg D_p = p^n$
* Prop (Step ②) $\Rightarrow$ $H_p = H_{\psi,p}$ is cocommutative for any homomorphism $\psi : R \to \overline{\mathbb{F}_p}$
* Com Alg $\Rightarrow$ direct product of all such ψ is an injection of R into a direct product of fields
* $\Rightarrow$ H_R is cocommutative
* $\Rightarrow$ H cocommutative & finite dim'l
* $\xRightarrow{\text{char } k = 0}$ $H \simeq kG$. ///

What's next?

$\rightsquigarrow$ Semisimple Hopf alg actions on quantizations of com. domains B (to be posted later)

In the meanwhile, we have a question:

The technique of the proof of the main result worked because for $B = A_n(k)$

★ $\to$ $\boxed{B_p \text{ (the reduction of } B \text{ modulo } p\text{) is a PI domain}}$

$\rightsquigarrow$ so we can localize to get division alg D_p.

★ $\to$ $\boxed{\text{PIdeg } B_p = p^n}$ for $p \gg 0$

$\rightsquigarrow$ so that we can employ Proposition on div. algs

But in general:

$\boxed{\text{Question}}$ Let B be a $\mathbb{N}$-filtered algebra over k with $gr(B)$ a commutative domain, finitely generated.

Does ★ hold for B, for any large prime p??